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New Spinor Fields on Lorentzian 7-Manifolds

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Abstract: This paper deals with the classification of spinor fields according to the bilinear covariants in 7 dimensions. The previously investigated Riemannian case is characterized by either one spinor field class, in the real case of Majorana spinors, or three non-trivial classes in the most general complex case. In this paper we show that by imposing appropriate conditions on spinor fields in 7d manifolds with Lorentzian metric, the formerly obtained obstructions for new classes of spinor fields can be circumvented. New spinor fields classes are then explicitly constructed. In particular, on 7-manifolds with asymptotically flat black hole background, these spinors can define a generalized current density which further defines a time Killing vector at the spatial infinity.

Contents

1 Introduction 1

2 General Bilinear Covariants and Spinor Field Classes in 7d 3

3 New Classes of 7d Spinors 5

4 Concluding Remarks and Outlook 8

A Appendix 9

1 Introduction

Classical spinor fields are characterized by their symmetry properties with respect to the rotation (Euclidean spacetime) or pseudorotation (pseudo-Euclidean spacetime) group. For instance in a 4d Minkowski geometry elementary spinors encompass two irreducible representations of the Lorentz group, the Weyl and Majorana spinors, and a reducible one, the Dirac spinors. With these spinors we can form bilinears, which in turn fully characterize the spinors themselves and satisfy the Fierz identities. If, on the other hand, we reverse the argument and assume that the defining properties of the spinors are the Fierz identities we find new surprising and (until recently) unexplored possibilities. The present paper is in the framework of this new field of research.

Fierz identities were used by Lounesto to classify spinor fields in Minkowski spacetime according to the bilinear covariants in six disjoint classes, that encompass all possible spinor fields in 4d Minkowski spacetime. All the spinor classes have been lately thoroughly characterized [1]. The first three classes of spinor fields in such classification are referred to as regular spinor fields. Their scalar and pseudo-scalar bilinear covariants are different from zero. The other three classes of singular spinors are called flag-dipole, flagpole and dipole spinor fields. In spite of including Weyl and Majorana spinors as very particular cases of dipole and flagpole spinors respectively [2], these new classes further contain genuinely new spinor fields with peculiar dynamics. For instance other flagpole spinor fields in these classes are eigenspinors of the charge conjugation operator with dual helicity and may be prime candidates for dark matter [3, 4]. Moreover, flag-dipole spinors were found to be solutions of the Dirac equation in Einstein-Sciama-Kibble (ESK) gravities [5]. A complete overview of this classification with further applications in field theory and gravitation can be found in [2]. This matter has been further explored in the context of black hole thermodynamics, where tunnelling methods were studied for the eigenspinors of the charge conjugation operator having dual helicity [15, 16], as special type of flagpoles [7]. Experimental signatures of type-5 spinors in Lounesto's classification may be related to the Higgs field at LHC [8].

Motivated by the new possibilities regarding such recently found new spinors, higher dimensional analogues have been recently introduced [9, 10], based upon the Fierz identities. In 7d there are plenty of examples in which new types of spinors may play a role and become relevant. Thus Fierz identities may provide an effective framework to attack problems in supergravity and string theory [10],[11, 12], in particular for what concerns compactifications of 11d SUGRA or M-theory to 7 or 4 dimensions. For a comprehensive review on the physical features related to the compactification procedure on S^7 see, e. g., [13]. Much in the same way as in the last decade new physical possibilities, beyond the standard Dirac, Majorana and Weyl spinors, have been introduced and studied in Minkowski 4d spacetime (see, e. g., Ref. [2] for a brief review), we aim here to establish the same bottom-up approach, providing the characterization of new classes of spinor fields on Lorentzian 7-manifolds.

In a previous paper we have constructed a classification of spinors in Riemannian 7-manifolds [9], based upon the fact that only some bilinears are different from zero [10]. The aim of the present paper is to show that, when an arbitrary spinor field in Lorentzian 7-manifolds is annihilated by a linear combination of the energy operator and the volume element, it singles out a new class of regular spinors.

It is remarkable that these spinors can be realized as soliton-like solutions in a specific black hole background and to identify the current density $J^\mu = \bar{\psi}\gamma^\mu\psi$ with the Killing vector at the black hole horizon [14]. This is inspired by Kerr and Myers-Perry 5d black holes, which constitutes an appropriate background wherein a current density interpolates between the time-like Killing vector field at the spatial infinity and the null Killing vector fielf on the black hole event horizon [14]. This current density can be realized as a spinor fluid flow. Here we consider the extension of this construction to 7d black holes. In fact, by imposing suitable conditions we show that the current-generating spinors, in a 7d black hole background in a Lorentzian 7-manifolds, precisely realises new classes of spinors, previously precluded by the Fierz identities on Euclidean 7-manifolds. In particular, the spinor class that we are going to study in Lorentzian signature has all the associated bilinear covariants different from zero. Hence these are the 7d analogues of regular spinors.

Finally, these regular spinors will be shown to further collapse into a specific class of singular ones, when the new regular spinor components satisfy specific constraints.

The classification we obtain in this paper is in no way exhaustive for 7d Lorentzian manifolds. Its main aim was to point out that under extremely simple algebraic conditions one can sort out a large class of regular spinors and construct them.

The paper is organized as follows: in section 2 the classification of spinor fields in Minkowski spacetime, according to the Lounesto's classification prescription, in 4d is reviewed. We also summarize the analogous one concerning Euclidean 7-manifolds, wherein complex spinors can be classified in three non-trivial classes, while the classification for real spin bundles encompasses Majorana spinor fields alone.

In section 3 the obstruction for the construction of other spinor fields on 7-manifolds is shown to be circumvented when a different spacetime signature is taken into account, by imposing certain conditions on the spinor fields. Subsequently we show that these new spinor fields can be explicitly constructed as soliton-like solutions in the framework of a

7d black-hole background. They are implicitly defined by a current of probability (which further defines the time-like Killing vector at the spatial infinity) and explicitly constructed via the reconstruction theorem. Comparing with the case of Riemannian 7-manifolds, we see that the number and types of spinor field classes, classified by the bilinear covariants via the Fierz identities, are signature dependent. Section 4 is devoted to our concluding remarks and outlooks.

2 General Bilinear Covariants and Spinor Field Classes in 7d

An oriented manifold (M, g) and its tangent bundle TM admits an exterior bundle $\bigwedge(TM)$. The Clifford product involving an arbitrary 1-form field $v \in \sec \bigwedge^1(TM)$ and an arbitrary form $a \in \sec \bigwedge(TM)$ is specified by a combination of the exterior product and the contraction, namely, $v \circ a = v \wedge a + v \lrcorner a$. By taking the particular case of Minkowski spacetime, the basis $\{e^\mu\}$ represents a section of the coframe bundle $P_{\mathrm{SO}^e_{1,3}}(M)$. Classical Dirac spinor fields are elements that carry the $\rho = (1/2, 0) \oplus (0, 1/2)$ representation of the Lorentz group. For any spinor field $\psi \in \sec P_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$, the bilinear covariants are given by:

$$\sigma = \bar{\psi}\psi\,, \tag{2.1a}$$

$$J_\mu e^\mu = \mathbf{J} = \bar{\psi}\gamma_\mu\psi\, e^\mu\,, \tag{2.1b}$$

$$S_{\mu\nu} e^\mu \wedge e^\nu = \mathbf{S} = \tfrac{1}{2} i\bar{\psi}\gamma_{\mu\nu}\psi\, e^\mu \wedge e^\nu\,, \tag{2.1c}$$

$$K_\mu e^\mu = \mathbf{K} = i\bar{\psi}\gamma_5\gamma_\mu\psi\, e^\mu\,, \tag{2.1d}$$

$$\omega = -\bar{\psi}\gamma_5\psi\,, \tag{2.1e}$$

where $\bar{\psi} = \psi^\dagger\gamma_0$, $\gamma_5 := \gamma_0\gamma_1\gamma_2\gamma_3$ and $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\mathbf{1}$. The Fierz identities read

$$\mathbf{K} \wedge \mathbf{J} = (\omega + \sigma\gamma_5)\mathbf{S}, \qquad \mathbf{J}^2 = \omega^2 + \sigma^2, \qquad \mathbf{K}^2 + \mathbf{J}^2 = 0 = \mathbf{J} \cdot \mathbf{K}\,. \tag{2.2}$$

When either $\omega \neq 0$ or $\sigma \neq 0$ [$\omega = 0 = \sigma$] the spinor field ψ is named regular [singular] spinor.

Lounesto classified spinor fields into six disjoint classes. In the classes (1), (2), and (3) beneath it is implicit that $\mathbf{J}$, $\mathbf{K}$ and $\mathbf{S}$ are simultaneously different from zero, and in the classes (4), (5), and (6) just $\mathbf{J} \neq 0$:

1) $\sigma \neq 0, \quad \omega \neq 0$

2) $\sigma \neq 0, \quad \omega = 0$

3) $\sigma = 0, \quad \omega \neq 0$

4) $\mathbf{K} \neq 0, \quad \mathbf{S} \neq 0 \quad (\sigma = \omega = 0)$

5) $\mathbf{K} = 0, \quad \mathbf{S} \neq 0 \quad (\sigma = \omega = 0)$

6) $\mathbf{S} = 0, \quad \mathbf{K} \neq 0 \quad (\sigma = \omega = 0)$

Singular spinor fields of types-4, -5, and -6 are flag-dipoles, flagpoles and dipole spinor fields, respectively. It is worth to emphasize that in classes (4), (5) and (6) the vectors $\{\mathbf{J}, \mathbf{K}\}$ can not be elements of a basis for Minkowski spacetime and collapse into a null-line. By defining $\mathbf{J}$ as the pole, flagpoles are hence defined in the class-(5), since for this case $\mathbf{K} = 0$ and $\mathbf{S} \neq 0$. The 2-form $\mathbf{S}$ is interpreted as a plaquette, namely, a flag-pole. In

the case of the type-4 spinor fields, both $\mathbf{S}$ *and* $\mathbf{K}$ are not equal zero, and they form a flag-dipole. Both concepts encompass Penrose flagpoles [6]. The first physical example of flag-dipole spinor fields has been recently found to be a solution of the Dirac equation in ESK gravities [5]. Moreover, Majorana and Elko spinor fields reside in the class of type-5 spinors, whereas Weyl spinor fields are a particular example of a type-6 dipole spinor fields, that further encompass pure spinors as well [2]. The characterization of new singular spinor fields in Lounesto's classes has introduced new fermions, including mass dimension one matter fields, that have been studied in [1–5, 7]. The most general types of spinor fields in each class of Lounesto's classification have been developed in [1]. For singular spinors the Fierz identities (2.2) read [17]:

$$\mathrm{Z}^2 = 4\sigma\mathrm{Z}, \quad \mathrm{Z}\gamma^\mu\mathrm{Z} = 4J^\mu\mathrm{Z}, \quad \mathrm{Z}i\gamma^5\gamma^\mu\mathrm{Z} = 4K^\mu\mathrm{Z}, \quad \mathrm{Z}i\gamma^{\mu\nu}\mathrm{Z} = 4S^{\mu\nu}\mathrm{Z}, \quad \mathrm{Z}\gamma^5\mathrm{Z} = -4\omega\mathrm{Z}\,, \tag{2.3}$$

where $\mathrm{Z} = \sigma + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \omega\gamma_{0123}$.

In arbitrary manifolds with (p, q) signature $p+q = n = \dim M$), given a spin bundle S and $\gamma^{n+1} \in \sec(\mathrm{End}(S))$ [10], spin projectors $\Pi_\pm = \frac{1}{2}(I \pm \gamma^{n+1})$ can be constructed. They provide the spin bundle splitting $S = S^+ \oplus S^-$, where $S^\pm = \Pi_\pm(S)$. Sections of $S^\pm$ are named Majorana-Weyl spinors when $p - q \equiv 0 \mod 8$, whereas sections of S^+ are known as Majorana spinors when $p - q \equiv 7 \mod 8$.

Let an orthonormal coframe be given, in 7d, by $\{e^a\}_{a=0}^6$. Hereupon we adopt the notation $\gamma_{\rho_1\rho_2\ldots\rho_k} = \gamma_{\rho_1}\gamma_{\rho_2}\cdots\gamma_{\rho_k}$ and $e^{\rho_1\cdots\rho_k} = e^{\rho_1} \wedge \cdots \wedge e^{\rho_k}$. In general the spinor conjugation reads $\bar\psi = \psi^\dagger a^{-1}$, for $a \in \mathcal{C}\ell_{p,q}^* = \mathcal{C}\ell_{p,q}\backslash\{0\}$ where $\mathcal{C}\ell_{p,q}$ denotes the Clifford bundle on a spacetime with signature (p, q). Given $\psi, \psi' \in \Gamma(S)$, and given a bilinear form B on $\Gamma(S)$, the most general bilinear on S read:

$$\beta_k(\psi, \psi') = B(\psi, \gamma_{\rho_1\ldots\rho_k}\psi') = \bar\psi\gamma_{\rho_1\ldots\rho_k}\psi'\,. \tag{2.4}$$

Now, generalized bilinear covariants are defined by [9, 10]

$$\varphi_k := \frac{1}{k!}B(\psi, \gamma_{\rho_1\cdots\rho_k}\psi)e^{\rho_1\cdots\rho_k} = \bar\psi\gamma_{\rho_1}\ldots\gamma_{\rho_k}\psi\, e^{\rho_1\cdots\rho_k} \in \sec\bigwedge^k(TM)\,. \tag{2.5}$$

For ψ a Majorana spinor, the forms φ_k equal zero except when either $k = 0$ or $k = 4$ [9, 10]. I.e. such class of Majorana spinor fields, according to the bilinears in the Clifford bundle $\mathcal{C}\ell_{7,0}$, is provided by:

$$\varphi_0 \neq 0, \quad \varphi_1 = 0, \quad \varphi_2 = 0, \quad \varphi_3 = 0, \quad \varphi_4 \neq 0, \quad \varphi_5 = 0, \quad \varphi_6 = 0, \quad \varphi_7 = 0\,. \tag{2.6}$$

The bilinear covariants in Eq.(2.5), except φ_0 and φ_4, were shown to be null on Euclidean 7-manifolds in Ref.[9], as a consequence of the geometric Fierz identities [10]. We will see in the next section that these obstructions can be circumvented in a different signature, once appropriate conditions are imposed. In other words, we obtain the bilinear covariants for the associated spinor fields when appropriate conditions are enforced on them [19]. More precisely, by imposing that the spinor field is annihilated by the one of the operators $\gamma^0 \pm \gamma^8$, we can prove that new distinct classes do exist in the above classification.

It is still worth to mention that real representations associated to the Clifford algebra over the (6,1) Lorentzian space admit a quaternionic structure induced by globally defined operators J_i $(i = 1, 2, 3)$ [10]. Such structure can be used to construct the following bilinear covariants:

$$\mathring{\varphi}_k := \frac{1}{k!} B(\psi, J_i \circ \gamma_{\rho_1 \ldots \rho_k} \psi) e^{\rho_1 \cdots \rho_k} := \mathring{\bar{\psi}} \gamma_{\rho_1} \ldots \gamma_{\rho_k} \psi \, e^{\rho_1 \cdots \rho_k} \in \sec \bigwedge^k (TM) \,. \tag{2.7}$$

Here the product "$\circ$" is the standard product in the spin bundle $\text{End}(S)$. In [10], for Euclidean 7-manifolds, Eq.(2.7) was proved to be related to (2.5) by Hodge duality. However it does not necessarily hold for the case of Lorentzian manifolds. In the latter case the quaternionic structure in (2.7) induced by the J_i can be taken into account. Hence there is a S^2-family of complex structures that can define an S^2 family of bilinear covariants $\mathring{\varphi}_k$. We remark, nevertheless, that this can be equivalently realized by incorporating the J_i in the conjugate $\bar{\psi}$ of the spinor ψ in Eq.(2.4), denoted by $\mathring{\bar{\psi}}$ in Eq. (2.7), clearly defining new equivalent spinor conjugates. A thorough discussion concerning quaternionic structures on manifolds of with different signatures can be found in [10].

3 New Classes of 7d Spinors

In this section we show that in a Lorentzian 7-manifold some obstructions found in the Euclidean case for the existence of general spinors can be circumvented. In fact we show that (analogues of) regular spinors can be effectively constructed.

Let us define a vielbein basis of the Clifford algebra over the (6,1) Lorentzian space as follows

$$\gamma^0 = i\sigma^1 \otimes \mathbb{I} \,, \quad \gamma^6 = \sigma^3 \otimes \mathbb{I} \,, \quad \gamma^a = -\sigma^2 \otimes \gamma_\diamond^{a-1} \,, \quad a = 1, \ldots, 5 \,, \tag{3.1}$$

where $\mathbb{I}$ denotes hereupon 4×4 identity operator and $\gamma_\diamond^\mu$ are provided by [19, 20]

$$\gamma_\diamond = i\sigma^1 \otimes \mathbf{1}_2 \,, \quad \gamma_\diamond^4 = \sigma^3 \otimes \mathbf{1}_2 \,, \quad \gamma_\diamond^j = -\sigma^2 \otimes \sigma^j \,, \quad j = 1, 2, 3 \tag{3.2}$$

The vielbein basis (3.1) is constructed from (3.2) by the method described in [20]. Exploiting the fact that bilinear covariants are representation-independent, (3.1) is chosen to provide a nicer form for the spinor components, see (3.3 - 3.5), under the condition (3.4) below.

Now the relevant spinor field is represented as

$$\psi = (\alpha_0, \ldots, \alpha_7)^\intercal \in \sec P_{\text{Spin}^e_{1,6}}(M) \times_\rho \mathbb{C}^8 \,, \tag{3.3}$$

where ρ stands for a representation of the associated Lorentz group and the α_a $(a=0,\ldots,7)$ are complex functions. The spinor ψ is required to satisfy the condition

$$(\gamma^0 \pm \gamma^8)\psi = 0 \,, \tag{3.4}$$

which yields [14]

$$\alpha_\mu = \alpha_{\mu+4}, \qquad \mu = 0,1,2,3 \,. \tag{3.5}$$

This condition is the only linear combination of gamma matrices (or products of gamma matrices) providing conditions for the spinor components that generate new classes of spinor fields.

By calculating the bilinear covariants in Eqs.(2.5) one can straightforwardly realize that all the bilinears are *generically* different from zero (unless very particular constraints among the spinor components hold. The conditions are derived in the Appendix.). In fact, for these kind of spinors, generically, we have

$$\varphi_0 \neq 0, \quad \varphi_1 \neq 0, \quad \varphi_2 \neq 0, \quad \varphi_3 \neq 0, \quad \varphi_4 \neq 0, \quad \varphi_5 \neq 0, \quad \varphi_6 \neq 0, \quad \varphi_7 \neq 0\,. \tag{3.6}$$

Hence, spinors (3.3) associated to the above bilinear covariants play the role of regular spinors on Lorentzian 7-manifolds. [1] In Appendix we explicitly calculate the bilinear covariants defined in (2.5) when the spinor (3.3) is subject to the condition (3.4). and we prove that all the bilinear covariants are different from zero unless very specific constraints are satisfied. Such possible exceptions are:

$$1): \ \varphi_0 = 0, \quad \text{if} \quad \alpha_2\overline{\alpha}_1 + \alpha_1\overline{\alpha}_2 + \alpha_4\overline{\alpha}_3 + \alpha_3\overline{\alpha}_4 = 0\,; \tag{3.7}$$

$$2): \ \varphi_6 = 0, \quad \text{if} \quad \begin{cases} \alpha_2\overline{\alpha}_1 - \alpha_1\overline{\alpha}_2 + \alpha_4\overline{\alpha}_3 - \alpha_3\overline{\alpha}_4 = 0\,, \\ |\alpha_1|^2 + |\alpha_2|^2 - |\alpha_3|^2 - |\alpha_4|^2 = 0 \end{cases} \tag{3.8}$$

$$3): \ \varphi_7 = 0, \quad \text{if} \quad \alpha_2\overline{\alpha}_1 - \alpha_1\overline{\alpha}_2 + \alpha_4\overline{\alpha}_3 - \alpha_3\overline{\alpha}_4 = 0\,. \tag{3.9}$$

In the particular case where the above conditions 1), 2), and 3) are simultaneously satisfied, by analysing the terms in (A.2), (A.8) and (A.9) and equating them to zero we get

$$\frac{|\alpha_4|^2\,|\alpha_3|^2}{|\alpha_3|^2 + |\alpha_4|^2} = |\alpha_2|^2\left(1 - |\alpha_2|^2\right)\,. \tag{3.10}$$

If this condition is satisfied, then $\varphi_0 = 0 = \varphi_6 = \varphi_7$.

A clarification is in order at this point. If no condition such as Eq.(3.4) is imposed on a generic spinor ψ, all bilinear covariants are generically nonvanishing. Indeed, if no condition is imposed on the spinor ψ, then the spinor components (3.3) are generic and will satisfy (3.6). Condition (3.4), however, assures the computational feasibility of finding physical solutions of the Dirac equation in Lorentzian manifolds. Since we show that the possible spinor classes are restricted to eight, instead of the 128 initial possible ones, Moreover, the spinor components are four linearly independent ones due to (3.5), instead of the initial eight ones. All this makes the quest for solutions much more manageable. It is obvious from the above that our aim in this paper is not to exhaust all the possibilities for spinor fields in 7d Lorentzian manifolds, but rather to select a subclass of them with remarkable properties, in particular such that they can be identified as regular spinors.

[1]The conditions (3.6) and the additional ones in Appendix are pointwise, so the question arises of such spinors being globally defined. The spacetime spin manifolds of major interest are Lorentzian simply connected manifold with trivial holonomy, that is maximally symmetric spaces (dS, AdS or Minkowski). In such cases the conditions (3.6) being true in a single chart is enough (see, for instance, the explicit example shown below). If the manifold M has a more complicated topology one has of course to check that the conditions are preserved by the relevant transition functions.

Among these properties let us cite also the difference with the case of 4d Minkowski spacetime. Here an arbitrary spinor (3.3) will trivially satisfy (3.6); however, the spinor (3.3) with components satisfying (3.5) still satisfy (3.6), a result that has no analogue in the standard Lounesto's 4d classification [2].

The spinor ψ can be explicitly constructed from the above bilinears. In fact, a spinor can be determined up to a phase from the bilinear covariants, by the inversion theorem [17]. Given ψ let us consider an aggregate [9]

$$\mathrm{Z} = \sum_{k=0}^{7} \bar{\psi}\gamma_{\rho_1\ldots\rho_k}\psi\, e^{\rho_1\cdots\rho_k}, \tag{3.11}$$

where the sum is ordered in k. Starting from an arbitrary 7d spinor τ satisfying $\widetilde{\tau^*}\psi \neq 0$, the original spinor ψ can be recovered from its aggregate (3.11). The spinor ψ and the multivector field[2] $\mathrm{Z}\tau$ are equivalent up to a scalar:

$$\psi = \frac{1}{2\sqrt{\tau^\dagger\gamma_0\mathrm{Z}\tau}}\, e^{-i\vartheta}\mathrm{Z}\tau, \tag{3.12}$$

where $e^{-i\vartheta} = 2(\tau^\dagger\gamma_0\mathrm{Z}\tau)^{-1/2}\tau^\dagger\gamma_0\psi \in \mathrm{U}(1)$. This generalizes to 7d the well known reconstruction theorem (Takahashi algorithm) [17], leading to the 4d equivalent of (3.12). It is worth to mention that a comprehensive discussion and a complete proof regarding the reconstruction theorem for more general cases is carried out in [21].

Next we would like to show that such types of new fermions may appear, as soliton-like solutions, in a suitable black hole background. The metric for stationary and axisymmetric black holes in 7d can be expressed by [14]

$$\begin{aligned} ds^2 &= -f_t(\Pi dt + f^i d\phi_i)^2 + f_r dr^2 + f_1^A d\theta_A^2 + g_{11}\Big[d\phi_1 - w_1 dt + g_{12}(d\phi_2 - w_2 dt) \\ &\quad + g_{13}(d\phi_3 - w_3 dt)\Big]^2 + g_{22}\Big[d\phi_2 - w_2 dt + g_{23}(d\phi_3 - w_3 dt)\Big]^2 + g_{33}(d\phi_3 - w_3 dt)^2, \end{aligned} \tag{3.13}$$

where $\Pi = 0$ or 1, $A = 1, 2$ and the functions f_a, g_{ab} and w_a depend only upon the radius r and the latitudinal angles θ^A. The latter can be always chosen to be positive definite near the black hole horizon [14]. In such cases $w^a \to \Omega^a$ as $r \to R$, [14] where R denotes a coordinate singularity and Ω^a stands for the angular velocity of the black hole in the ϕ^a direction. Hence (3.13) can be rewritten in terms of vielbeins

$$ds^2 = \eta_{AB}e^A e^B\,, \quad A, B = 0, \ldots, 6\,, \tag{3.14}$$

where $\eta = \mathrm{diag}\,(- + \cdots +)$. In general, (3.14) is only well defined near the black hole horizon, where all the vielbeins are real [14].

Now, using an arbitrary spinor satisfying (3.4), let us consider the vector field

$$\xi^\rho = b_\psi \bar{\psi}\gamma^\rho\psi\,. \tag{3.15}$$

[2] It is worth to mention that $\mathrm{Z}\tau$ is, a priori, a multivector that we can prove to be an element of a minimal left ideal in the associated Clifford algebra. Hence it is in fact an algebraic spinor, according to the Chevalley construction.

In the particular case where $b_\psi = 1$, this is the current density $J^\rho = \bar{\psi}\gamma^\rho\psi$. In the most general case, b_ψ is some scalar. The spinor ψ is not necessarily a regular fermion [9], and in particular it can be the higher dimensional version of flag-dipoles, flagpoles or dipoles singular spinors fields [2]. Given that the spinor field ψ obeys one of the two conditions (3.4), the current density $J^\rho = \bar{\psi}\gamma^\rho\psi$ reduces to the expression

$$J^\rho\partial_\rho = 2\left(|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2\right)\partial_t \,. \tag{3.16}$$

Although this is outside the topic of this paper we notice that, since w^a are constants on the horizon [14], the vector field $\xi = \xi^\rho\partial_\rho$ in Eq. (3.15) – a multiple of the current density in Eq. (3.16) – can be identified with the null Killing vector on the black hole horizon. In fact the vector field ξ can be shown to interpolate between the time Killing vector at the spatial infinity and the null Killing vector on the horizon [19]. In addition, $\nabla_\mu\xi^\mu = 0$, what justifies calling ξ a conserved current.

A 1-form current density $\xi^\rho = b_\psi\bar{\psi}\gamma^\rho\psi$ has been constructed, being the conserved current of a particular spinor field. In the background of a stationary black hole the current density vector field always approaches the null Killing vector at the horizon. When the black hole is asymptotically flat and when the coordinate system is asymptotically static, the same vector field also becomes the time Killing vector at spatial infinity [19]. The required constraint on the spinor field unblock the obstructions that the Fierz identities impose on new classes of spinors fields. In fact we have proved that, in the context of this paper, the bilinear covariants (2.5) are different from zero. It is worth to emphasize that when the above mentioned coefficient b_ψ is non-vanishing, we can view the left hand side of Eq.(3.16) as a well-defined Killing vector.

We recall that these results, different from the ones obtained in [9], are explained by the fact that the metric used here is Lorentzian, while in [9] it was Euclidean.

4 Concluding Remarks and Outlook

The geometric Fierz identities [9, 10] are well known to limit the number of classes of spinor fields according to the bilinear covariants. In a previous paper we have investigated Majorana spinor fields in Euclidean 7-manifolds, proving that the geometric Fierz identities forbid the existence of more than one spinor field class (3.6) in the real case [9], while three non-trivial classes can exist in the complex case. However the obstructions that preclude the existence of further spinor field classes in Euclidean 7-manifolds can be attenuated in a different spacetime metric signature, when conditions are imposed on the spinor fields.

We have achieved this by imposing one of the conditions (3.4), for in this case the associated spinor field have all bilinear covariants non-vanishing. This property has remarkable implications. In Lounesto's spinors classification in 4d Minkowski spacetime, the bilinear covariants for spinors of type-1 are all non zero, and regular spinors play the role of the standard Dirac spinor, describing, for instance, the electron in the Dirac theory. Here, in a 7-manifold with Lorentzian signature, calculating the bilinear covariants for a spinor under the condition (3.4), we prove that all bilinear covariants are generically non-vanishing, We deduce that in a generalized classification of spinors in Lorentzian 7-manifolds according to

the observables (2.5), the spinor (3.3) with the constraint (3.4) can be interpreted as the analogues of the regular spinor in 4d.

As we have remarked, an exception is when the constraint (3.10) holds. In this case we have a restriction $\varphi_k = 0$, for either $k = 0$ or $k = 6$ or $k = 7$. However, preliminary calculations show that some of the conditions (3.7, 3.8, and 3.9) can be only achieved for spinors that are solutions of the Dirac equation in Lorentzian 7-manifolds with torsion. Hence, without torsion (or a Kalb-Ramond background field), our results show that there is only one class of regular spinors.

It should also be recalled that we have studied the classification of spinors related to the standard bilinear covariants based upon (2.5). One could also consider an S^2 family of isomorphic classifications provided by the quaternionic structures J_i in (2.7). These cases are already included in the previous analysis for we can always incorporate the representations of the J_i in the conjugate spinor in (2.7). Since the bilinear covariants are representation independent, the spinor classifications induced via the quaternionic structures are indeed isomorphic to those obtained above.

Once the classes of spinor fields on both Euclidean and Lorentzian 7-manifolds have been identified, a further question arises by considering inequivalent spin structures on 7-manifolds. They are well known to induce an additional term on the associated Dirac operator, related to a Čech cohomology class [3, 22]. Although the existence and classification in the above sections is not modified by considering inequivalent spin structures, the dynamics associated to spinor fields in each class can manifest important modifications, which we shall discuss in a forthcoming publication.

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A Appendix

When the spinor (3.3) under the condition (3.4) is taken into account, we can explicitly calculate the bilinear covariants defined in (2.5) as:

$$\varphi_0 = 2i\left(\alpha_2\overline{\alpha}_1 + \alpha_1\overline{\alpha}_2 + \alpha_4\overline{\alpha}_3 + \alpha_3\overline{\alpha}_4\right) \tag{A.1}$$

$$\begin{aligned} \varphi_1 &= \bar{\psi}\gamma_k\psi\, e^k \\ &= 2\left(|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2\right) e^0 - 2\left(|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2\right) e^5 \end{aligned} \tag{A.2, A.3}$$

It implies that unless $|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 = 0$, namely, when $\alpha_1 = 0 = \cdots = \alpha_4$, the 1-form bilinear covariant is always different from zero.

Now, the 2-form bilinear covariant explicitly reads:

$$\begin{aligned}
\varphi_2 &= \bar{\psi}\gamma_{k_1k_2}\psi \; e^{k_1k_2} \\
&= 2i\left(\alpha_2\overline{\alpha}_1+\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3+\alpha_3\overline{\alpha}_4\right)\; e^{05} - 2\left(\alpha_2\overline{\alpha}_1-\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3-\alpha_3\overline{\alpha}_4\right)\; e^{12} \\
&\quad -2i\left(|\alpha_1|^2-|\alpha_2|^2+|\alpha_3|^2-|\alpha_4|^2\right)\; e^{13} + 2\left(|\alpha_1|^2-|\alpha_2|^2+|\alpha_3|^2-|\alpha_4|^2\right)\; e^{14} \\
&\quad +2i\left(|\alpha_1|^2-|\alpha_2|^2+|\alpha_3|^2-|\alpha_4|^2\right)\; e^{16} + 2\left(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2\right)\; e^{23} \\
&\quad +2i\left(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2\right)\; e^{24} - 2\left(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2\right)\; e^{26} \\
&\quad +2\left(\alpha_2\overline{\alpha}_1+\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3+\alpha_3\overline{\alpha}_4\right)\; e^{34} + 2i\left(\alpha_2\overline{\alpha}_1+\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3+\alpha_3\overline{\alpha}_4\right)\; e^{36} \\
&\quad -2\left(\alpha_2\overline{\alpha}_1+\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3+\alpha_3\overline{\alpha}_4\right)\; e^{56}
\end{aligned} \tag{A.4}$$

Since the above spinor components α_μ are functions of the point in the manifold, then all terms of the above 2-form must be zero in order for φ_2 to vanish. It occurs if and only if all $\alpha_\mu = 0$. Therefore $\varphi_2 \neq 0$ for all non-trivial spinor $\psi \in \sec P_{\mathrm{Spin}^e_{1,6}}(M) \times_\rho \mathbb{C}^8$ under the condition (3.4).

Next let us display the 3-form bilinear covariant:

$$\begin{aligned}
\varphi_3 &= \bar{\psi}\gamma_{k_1k_2k_3}\psi \; e^{k_1k_2k_3} \\
&= 2i\left(|\alpha_1|^2-|\alpha_2|^2+|\alpha_3|^2-|\alpha_4|^2\right)\; e^{012} - 2\left(\alpha_2\overline{\alpha}_1-\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3-\alpha_3\overline{\alpha}_4\right)\; e^{013} \\
&\quad -2i\left(\alpha_2\overline{\alpha}_1-\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3-\alpha_3\overline{\alpha}_4\right)\; e^{014} + 2\left(\alpha_2\overline{\alpha}_1-\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3-\alpha_3\overline{\alpha}_4\right)\; e^{016} \\
&\quad +2i\left(\alpha_2\overline{\alpha}_1-\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3-\alpha_3\overline{\alpha}_4\right)\; e^{023} - 2\left(\alpha_2\overline{\alpha}_1-\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3-\alpha_3\overline{\alpha}_4\right)\; e^{024} \\
&\quad -2i\left(\alpha_2\overline{\alpha}_1-\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3-\alpha_3\overline{\alpha}_4\right)\; e^{026}2i\left(|\alpha_1|^2-|\alpha_2|^2+|\alpha_3|^2-|\alpha_4|^2\right)\; e^{034} \\
&\quad -2\left(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2\right)\; e^{036} - 2i\left(|\alpha_1|^2-|\alpha_2|^2+|\alpha_3|^2-|\alpha_4|^2\right)\; e^{056} \\
&\quad +2i\left(|\alpha_1|^2-|\alpha_2|^2+|\alpha_3|^2-|\alpha_4|^2\right)\; e^{126} - 2i\left(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2\right)\; e^{245} \\
&\quad +2i\left(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2\right)\; e^{256} - 2i\left(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2\right)\; e^{345} \\
&\quad +2\left(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2\right)\; e^{356}+2i\left(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2\right)\; e^{456}
\end{aligned} \tag{A.5}$$

All terms of φ_3 have to be zero in order to φ_3 to be zero. Indeed, in particular the coefficient $\left(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2\right)$ of e^{356} must be zero in order to $\varphi_4 = 0$, which implies that $\alpha_\mu = 0$. Hence $\varphi_3 \neq 0$ for all non-trivial spinor $\psi \in \sec P_{\mathrm{Spin}^e_{1,6}}(M) \times_\rho \mathbb{C}^8$ under the condition (3.4).

The 4-form bilinear covariant is

$$\begin{aligned}
\varphi_4 &= \bar{\psi}\gamma_{k_1k_2k_3k_4}\psi \; e^{k_1k_2k_3k_4} \\
&= 2\left(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2\right)\; e^{0235} + 2\left(\alpha_2\overline{\alpha}_1+\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3+\alpha_3\overline{\alpha}_4\right)\; e^{0345} \\
&\quad +2\left(\alpha_2\overline{\alpha}_1+\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3+\alpha_3\overline{\alpha}_4\right)\; e^{0356} + 2\left(\alpha_2\overline{\alpha}_1+\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3+\alpha_3\overline{\alpha}_4\right)\; e^{0456} \\
&\quad -2\left(\alpha_2\overline{\alpha}_1+\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3+\alpha_3\overline{\alpha}_4\right)\; e^{1236} - 2i\left(\alpha_2\overline{\alpha}_1-\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3-\alpha_3\overline{\alpha}_4\right)\; e^{1246} \\
&\quad 2\left(|\alpha_1|^2-|\alpha_2|^2+|\alpha_3|^2-|\alpha_4|^2\right)\; e^{1346}+2\left(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2\right)\; e^{2346}
\end{aligned} \tag{A.6}$$

Again, the coefficient $\left(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2\right)$ of e^{2346} must be zero in order for $\varphi_5=0$, implying that $\alpha_\mu=0$. Hence $\varphi_4\neq 0$ for all non-trivial spinor $\psi\in\sec P_{\mathrm{Spin}^e_{1,6}}(M)\times_\rho\mathbb{C}^8$ under the condition (3.4).

The 5-form one is

$$\begin{aligned}
\varphi_5 &= \bar\psi\gamma_{k_1k_2\ldots k_5}\gamma_8\, e^{k_1k_2\ldots k_5}\\
&= 2\left(|\alpha_1|^2-|\alpha_2|^2+|\alpha_3|^2-|\alpha_4|^2\right)\ e^{01234}+2i+2\left(\alpha_2\overline{\alpha}_1-\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3-\alpha_3\overline{\alpha}_4\right)\ e^{02346}\\
&\quad -2\left(|\alpha_1|^2-|\alpha_2|^2+|\alpha_3|^2-|\alpha_4|^2\right)\ e^{02346}\\
&\quad +2\left(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2\right)\ e^{12345}+2i\left(|\alpha_1|^2-|\alpha_2|^2+|\alpha_3|^2-|\alpha_4|^2\right)\ e^{12356}\\
&\quad +2\left(\alpha_2\overline{\alpha}_1+\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3+\alpha_3\overline{\alpha}_4\right)\ e^{23456}
\end{aligned}\tag{A.7}$$

With respect to the above equation, we want to analyze in which cases we have $\varphi_5=0$. Let us then take, in particular, the coefficient of e^{12345} to be zero, which implies that $\alpha_\mu=0$. Hence there is no non-trivial spinor leading to such situation.

A similar reasoning can be applied to the 6-form below:

$$\begin{aligned}
\varphi_6 &= \bar\psi\gamma_{k_1}\gamma_8\, e^{k_1\ldots k_6}\\
&= -2i\left(\alpha_2\overline{\alpha}_1-\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3-\alpha_3\overline{\alpha}_4\right)\ e^{012346}-2\left(|\alpha_1|^2+|\alpha_2|^2-|\alpha_3|^2-|\alpha_4|^2\right)\ e^{123456}\\
&\quad -2\left(|\alpha_1|^2+|\alpha_2|^2-|\alpha_3|^2-|\alpha_4|^2\right)e^{013456}+2i\left(\alpha_2\overline{\alpha}_1-\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3-\alpha_3\overline{\alpha}_4\right)\ e^{012456}
\end{aligned}\tag{A.8}$$

Finally we calculate the 7-form:

$$\varphi_7=2i\left(\alpha_2\overline{\alpha}_1-\alpha_1\overline{\alpha}_2+\alpha_4\overline{\alpha}_3-\alpha_3\overline{\alpha}_4\right)\ e^{012456}\tag{A.9}$$

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